

Generalized Friedmann Equations for a Finite Thick Brane

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Abstract

We present the generalized Friedmann equations describing the cosmological evolution of a finite thick brane immersed in a five-dimensional Schwarzschild Anti-de Sitter spacetime. A linear term in the density in addition to a quadratic one arises in the Friedmann equation, leading to the standard cosmological evolution at late times without introducing an ad hoc tension term for the brane. The effective four-dimensional cosmological constant is then uplifted similar to the KKLT effect and vanishes for a brane thickness equal to the AdS curvature size, up to the third order of the thickness. The four-dimensional gravitational constant is then equal to the five-dimensional one divided by the AdS curvature radius, similar to that derived by dimensional compactification. An accelerating brane cosmology may emerge at late times provided there is either a negative transverse pressure component in the brane energy-momentum tensor or the effective brane cosmological constant is positive.

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1 Introduction

An infinitesimally thin brane is a geometrical construction which may reflect the characteristic features of the solitonic objects found in string theory at the energy scales much smaller than the energy scales related to the inverse thickness of the brane, or at distance scales much bigger than the thickness of the brane. However, once we are going to use the concept of brane phenomenologically and apply it to cosmology we have to be careful about the interplay of different scales inherent in a cosmological model. For example, we have to be sure about the smallness of the effect of thickness on the density or the cosmological parameter before deciding to ignore it. This has never been shown explicitly.

Interest in walls and branes as solitonic localized matter distributions, specially in higher dimensions, came from string theory mainly because it provides a novel approach for resolving the cosmological constant and the hierarchy problems [1]. In this scenario, gravitation is localized on a brane reproducing effectively four-dimensional gravity at large distances due to the warp geometry of the spacetime [2]. However, the history of the interest in the localized matter distributions in the context of gravity goes back to the early beginning of general relativity. Recognizing the difficulty of handling thick walls within relativity, already early authors considered the idealization of a singular hypersurface as a thin wall and tried to formulate its dynamics within general relativity [3]. Einstein and Strauss used implicitly the concept of a thick shell to an embedded spherical star within a Friedmann-Robertson-Walker universe [4]. The new era of intense interests in thin shells and walls began with the development of ideas related to phase transitions in early universe and the formation of topological defects. Again, mainly because of technical difficulties, strings and domain walls were assumed to be infinitesimally thin [5].

Thereafter, interest in thin walls, or hypersurfaces of discontinuity, received an impetus from the cosmology of early universe. The formulation of dynamics of such singular hypersurfaces was summed up in the modern terminology by Israel [6]. Within the Sen-Lanczos-Israel (SLI) formalism, thin shells are regarded as idealized zero thickness objects, with a δ -function singularity in their energy-momentum and Einstein tensors.

In contrast to thin walls, thickness brings in new subtleties, depending on how the thickness is defined and handled. Early attempts to formulate thickness, being mainly motivated by the outcome of the idea of late phase transition in cosmology [7], were concentrated on domain walls. Silveria [8] studied the dynamics of a spherical thick domain wall by appropriately defining an average radius $\langle R \rangle$, and then used the well-known plane wall scalar field solution as the first approximation to derive a formula relating $\langle \ddot{R} \rangle$, $\langle \dot{R} \rangle$, and $\langle R \rangle$ as the equation of motion for the thick wall. Widrow [9] used the Einstein-scalar equations for a static thick domain wall with planar symmetry. He then took the zero-thickness limit of his solution and showed that the orthogonal components of the energy-momentum tensor would vanish in that limit. Garfinkle and Gregory [10] presented a modification of the Israel thin shell equations by using an expansion of the coupled Einstein-scalar field equations describing the thick gravitating wall in powers of the thickness of the domain wall around the well-known solution of the hyperbolic tangent kink for a $\lambda\phi^4$ wall and concluded that the effect of thickness at first approximation was effectively to reduce the energy density of the wall compared to the thin case, leading to a faster collapse of a spherical wall in vacuum. Others [11] applied the expansion in the wall action and integrate it out perpendicular to the wall to show that the effective action for a thick domain wall in vacuum apart, from the usual Nambu term, consists of a contribution proportional to

the induced Ricci curvature scalar.

Study of thick branes in the string inspired context of cosmology began almost simultaneously with the study of thin branes, using different approaches. Although in brane cosmology the interest is in local behavior of gravity and the brane, most of the authors take a planar brane for granted [12]. However, irrespective of the spacetime dimension and the motivation of having a wall or brane, as far as the geometry of the problem is concerned, most of the papers are based on a regular solution of Einstein equations on a manifold with specified asymptotic behavior representing a localized scalar field [13]. Some authors use a smoothing or smearing mechanism to modify the Randall-Sundrum ansatz [14, 15]. Authors in [15] introduce a thickness to the brane by smoothing out the warp factor of a thin brane world to investigate the stability of a thick brane. In another approach to derive generalized Friedmann equations, the four-dimensional effective brane quantities are obtained by integrating the corresponding five-dimensional ones along the extra-dimension over the brane thickness [16]. These cosmological equations describing a brane of finite thickness interpolate between the case of an infinitely thick brane corresponding to the familiar Kaluza-Klein picture and the opposite limit of an infinitely thin brane giving the unconventional Friedmann equation, where the energy density enters quadratically. The latter case is then made compatible with the conventional cosmology at late times by introducing and fine tuning a negative cosmological constant in the bulk and an intrinsic positive tension in the brane [17]. Recently, Navarro and Santiago [18] considered a thick codimension 1 brane including a matter pressure component along the extra dimension in the energy-momentum of the brane. By integrating the 5D Einstein equations along the fifth dimension, while neglecting the parallel derivatives of the metric in comparison with the transverse ones, they write the equations relating the values of the first derivatives of the metric at the brane boundary with the integrated components of the brane energy-momentum tensor. These, so called matching conditions are then used to obtain the cosmological evolution of the brane which is of a non-standard type, leading to an accelerating universe for special values of the model parameters.

A completely different approach based on the gluing of a thick wall considered as a regular manifold to two different manifolds on both sides of it was first suggested in [19]. The idea behind this suggestion is to understand the dynamics of a localized matter distribution of any kind confined within two metrically different spacetimes or matter phases. Such a matching of three different manifolds is envisaged to have many diverse applications in astrophysics, early universe, and string cosmology. It enables one to have any topology and any spacetime on each side of the thick wall or brane. The range of its applications is from the dynamics of galaxy clusters and their halos to branes in any spacetime dimension with any symmetry on each side of it [20]. By construction, such a matching is regular and there is no singular surface whatsoever in this formulation. Therefore Darmois junction conditions for the extrinsic curvature tensors on the thick wall boundaries with the two embedding spacetimes can be applied.

In this paper, we will use this formalism recently developed in [21] for a finite thick wall and apply it, as an example, to a thick brane embedded in a Schwarzschild Anti-de Sitter (Sch-AdS) bulk to see the effect of thickness on the cosmology of the brane. Although the dynamical equations can be written in an exact form, to compare them with the standard cosmology we have to make an expansion in terms of the brane thickness. It turns out that the modified Friedmann equations are similar to the standard one having a linear term in density, in contrast to the thin brane cosmology, where the density term enters quadratically.

In section 2 we introduce our formalism to examine the cosmological evolution of a thick brane

embedded in the bulk spacetime. Section 3 is devoted to the metric of the bulk and the brane and the related quantities needed to be substituted in the junction conditions. In section 4 we give the modified Friedmann equations for the thick brane. Our conclusions are presented in section 5.

Throughout the paper we use Λ for the five-dimensional cosmological constant and κ for its gravitational constant. The two boundary limits of the thick brane are called Σ_j with $j = 1, 2$. The core of the thick brane is denoted by Σ_0 . For any quantity S let S_0 denote $S|_{\Sigma_0}$. Square bracket $[F]$ indicates the jump of any quantity F across Σ_j . Latin indices range over the intrinsic coordinates of Σ_j denoted by ξ_j^a , and Greek indices over the coordinates of the 5-manifolds.

2 Modelling the Thick Brane

The technology of manipulating thin and thick localized matter distributions or walls in general relativity in any dimension are basically different. Thin walls can be treated in two different but equivalent ways. Either one solves the Einstein equations in $d+1$ dimension with a *distributional* energy-momentum tensor which mimics an infinitesimally thin wall carrying some kind of matter, dark or not dark, including radiation, or one takes the known solutions of Einstein equations on either side of the wall and glues them to the wall by applying the boundary conditions at the wall location. The equivalence of these two procedures is not trivial but has been shown rigorously for the general case in [22]. Boundary surfaces not carrying any energy-momentum tensor can just be considered as a special case. It should be noted that such an equivalence does not exist for codimension 2 walls or defects, as it is also the case for the strings in 4-dimensional spacetime.

The lack of such an equivalence in the case of thick walls or localized matter distributions makes us differentiate between different applications of the term thick walls or branes. Usually a thick wall is considered to be a solution of Einstein equations with a localized scalar field having a well-defined asymptotic behavior. As mentioned in the introduction, we will continue to use the term thick wall for such a solution of Einstein equations. However, there is another case of interest with advantages in diverse applications in astrophysics and string cosmology. Assume a localized matter distribution to be considered as a solution of Einstein equations on a specific manifold with well-defined timelike boundaries. This localized or thick wall is then immersed in a universe which could in principle consist of two different solutions of Einstein equations on each side of the wall. The combined manifold, consisting of three different solutions of Einstein equations is again a solution of Einstein equations. The localized wall may be infinite in the planar or cylindrical case or compact in the spherical case. Therefore, in contrast to the thin wall formalism where one glues two different manifolds along a *singular* hypersurface, our definition of localized or thick wall leads to the problem of gluing three different manifolds along two *regular* hypersurfaces.

Let us now consider a thick codimension 1 brane immersed in a 5-dimensional bulk spacetime. Following the formalism introduced in [21], we take the thick wall with two boundaries Σ_1 and Σ_2 dividing the overall spacetime \mathcal{M} into three regions. Two regions \mathcal{M}_+ and \mathcal{M}_- on either side of the wall and the region \mathcal{M}_0 within the wall itself. Treating the two surface boundaries Σ_1 and Σ_2 separating the manifold \mathcal{M}_0 from two distinct manifolds \mathcal{M}_+ and \mathcal{M}_- , respectively, as nonsingular timelike hypersurfaces, we expect the intrinsic metric h_{ab} and extrinsic curvature

tensor K_{ab} of Σ_j to be continuous across the corresponding hypersurfaces. These requirements, the so-called Darmois conditions, are formulated as

$$[h_{ab}]_{\Sigma_j} = 0 \quad j = 1, 2, \quad (1)$$

$$[K_{ab}]_{\Sigma_j} = 0 \quad j = 1, 2, \quad (2)$$

where the square bracket denotes the jump of any quantity that is discontinuous across Σ_j . To impose the Darmois conditions on two surface boundaries of a given thick wall one needs to know the metric in three distinct spacetimes \mathcal{M}_+ , \mathcal{M}_- and \mathcal{M}_0 being jointed at Σ_j . While the metrics in \mathcal{M}_+ and \mathcal{M}_- are usually given in advance, knowing the metric within the wall spacetime \mathcal{M}_0 requires a nontrivial work.

We assume the wall to have a proper thickness $2w$ in the fifth dimension. We therefore have three different regions, two in the bulk and one within the brane, to be joined together as three different manifolds. Each boundary of the thick brane (Σ_j , $j = 1, 2$) glues the inside metric to a version of the outside spacetime. From the two junction conditions for each of the boundaries, to glue a slicing of the bulk to the spacetime within the brane, we obtain:

$$K_{ab}|_{\Sigma_2}^+ - K_{ab}|_{\Sigma_1}^- + K_{ab}|_{\Sigma_1}^w - K_{ab}|_{\Sigma_2}^w = 0, \quad (3)$$

where $+$, $-$ denote two slices of the outside spacetime and w denotes the spacetime within the wall.

Now, we introduce a Gaussian normal coordinate system (n, ξ_0^a) in the neighborhood of the core of the thick brane denoted by Σ_0 , where ξ_0^a are the intrinsic coordinates of Σ_0 , and n is the proper length along the geodesics orthogonal to Σ_0 such that $n = 0$ corresponds to Σ_0 . Assuming that the brane thickness is small in comparison with its curvature radius, we then expand the extrinsic curvature tensor terms in the equation (3) in a Taylor series around Σ_0 situated at $n = 0$

$$K_{ab}|_{\Sigma_i} = K_{ab}|_{\Sigma_0} + \epsilon_j w \frac{\partial K_{ab}}{\partial n}|_{\Sigma_0} + O(w^2), \quad (4)$$

where $\epsilon_1 = -1$ and $\epsilon_2 = +1$. The derivative of the extrinsic curvature of the brane is related to the 5D geometric quantities as follows:

$$\frac{\partial K_{ab}}{\partial n} = K_{ad}K_b^d - R_{\mu\alpha\nu\sigma}n^\alpha n^\sigma e_a^\mu e_b^\nu, \quad (5)$$

with n^μ being the normal vector field to the brane, $R_{\mu\alpha\nu\sigma}$ the five-dimensional Riemann curvature tensor and $e_a^\mu = \frac{\partial x^\mu}{\partial \xi_0^a}$ are the four basis vectors tangent to the brane. Substituting (5) and (4) into (3) we arrive at the following equation being written on the core of the thick brane

$$\begin{aligned} K_{ab}|_{\Sigma_0}^+ - K_{ab}|_{\Sigma_0}^- &+ w \left((K_{ac}K_b^c - R_{\mu\sigma\nu\lambda}e_a^\mu e_b^\nu n^\sigma n^\lambda)|_{\Sigma_0}^- + (K_{ac}K_b^c - R_{\mu\sigma\nu\lambda}e_a^\mu e_b^\nu n^\sigma n^\lambda)|_{\Sigma_0}^+ \right. \\ &\left. - 2(K_{ac}K_b^c - R_{\mu\sigma\nu\lambda}e_a^\mu e_b^\nu n^\sigma n^\lambda)|_{\Sigma_0}^w \right) = 0. \end{aligned} \quad (6)$$

Having specified the metric of the bulk and the metric within the wall, all terms in the equation (6) are known and, therefore, the dynamics of the thick brane is given. In the limit ($w \rightarrow 0$) we should also reproduce the familiar thin wall equations.

3 Explicit calculations on the core

Armed with Eq. (6) we now proceed to study the dynamics of a thick brane of constant spacial curvature embedded in a five-dimensional negative bulk cosmological constant. According to the generalized Birkhoff's theorem the five-dimensional vacuum cosmologically symmetric solution of Einstein's equations is necessarily static and corresponds to Sch-AdS metric given by

$$ds^2 = -f(r)dT^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \quad (7)$$

with

$$f(r) = k - \frac{\Lambda}{6}r^2 - \frac{C}{r^2}, \quad (8)$$

where $d\Omega_k^2$ is the metric of the 3D hypersurfaces Σ of constant curvature that is parameterized by $k = 0, \pm 1$; Λ is the bulk cosmological constant, and constant C is identified with the mass of a black hole located at $r = 0$. Since we need to know the spacetime of the thick brane itself we here take the following ansatz for the metric of the brane being written in a Gaussian normal coordinate system in the vicinity of the core of the thick brane situated at $y = 0$

$$ds^2 = -n^2(t, y)dt^2 + dy^2 + a^2(t, y)d\Omega_k^2, \quad (9)$$

where $n^2(t, y)$ and $a^2(t, y)$ are some unknown functions to be determined by solving the Einstein equations within the brane with a suitable energy-momentum tensor, and y is the normal coordinate of the extra dimension. Compatibility with the cosmological symmetries requires that the energy-momentum tensor of the matter content in the brane takes the simple form

$$T_\nu^\mu = (-\rho, P_L, P_L, P_L, P_T), \quad (10)$$

where the energy density ρ , the longitudinal pressure P_L , and the transverse pressure P_T are functions of t and y .

Reading the metric (9) as $ds^2 = dy^2 + \gamma_{ab}(x^c, y)dx^a dx^b$, we now expand it in a Taylor series in the vicinity of the core Σ_0 of the thick brane placed at $y = 0$ as follows:

$$\gamma_{ab}(x^c, y) = \gamma_{ab}(x^c, 0) + y \frac{\partial \gamma_{ab}(x^c, y)}{\partial y} \Big|_{y=0} + \frac{y^2}{2} \frac{\partial^2 \gamma_{ab}(x^c, y)}{\partial y^2} \Big|_{y=0} + O(y^3), \quad (11)$$

where $\gamma_{ab}(x^c, 0)$ is the metric on Σ_0 , with $x^c = (\tau, \chi, \theta, \varphi)$ the intrinsic coordinates of Σ_0 . But the derivatives in the expansion (11) are given by

$$\frac{\partial \gamma_{ab}(x^\sigma, y)}{\partial y} = 2K_{ab}, \quad (12)$$

$$\frac{\partial^2 \gamma_{ab}(x^\sigma, y)}{\partial y^2} = 2K_{ad}K_b^d - 2R_{\mu\alpha\nu\sigma}n^\alpha n^\sigma e_a^\mu e_b^\nu. \quad (13)$$

Let us write it more explicitly

$$-n^2(t, y) = -n^2(t, 0) + 2yK_{\tau\tau} \Big|_{y=0} + y^2(K_\tau^\tau K_{\tau\tau} - R_{tyty}) \Big|_{y=0}, \quad (14)$$

$$a^2(t, y) = a^2(t, 0) + 2yK_{\chi\chi} \Big|_{y=0} + y^2(K_\chi^\chi K_{\chi\chi} - R_{\chi y \chi y}) \Big|_{y=0}. \quad (15)$$

Defining $a_0(\tau) = a(t(\tau), 0)$ and using the expansions (14) and (15), we now write down the non-trivial components of the full 5D Einstein's equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ at the location of the core of the brane placed at $y = 0$, with the metric (9) and the energy-momentum tensor (10) as follows (see the Appendix for the corresponding components of the Einstein tensor)

$$ty : -H_0 K_{\tau\tau} + \frac{H_0}{a_0^2} K_{\chi\chi} - \frac{1}{a_0^2} \dot{K}_{\chi\chi} = 0, \quad (16)$$

$$yy : \frac{K_{\chi\chi}^2}{a_0^4} - \frac{K_{\chi\chi} K_{\tau\tau}}{a_0^2} - H_0^2 - \frac{\ddot{a}_0}{a_0} - \frac{k}{a_0^2} = \frac{\kappa^2}{3} P_T^0, \quad (17)$$

$$tt : H_0^2 - \frac{1}{a_0^2} (B - k) = \frac{\kappa^2}{3} \rho_0, \quad (18)$$

$$\chi\chi : \frac{2\ddot{a}_0}{a_0} + \frac{2K_{\tau\tau} K_{\chi\chi}}{a_0^2} + \frac{K_{\chi\chi}^2}{a_0^4} + K_{\tau\tau}^2 - \frac{B}{a_0^2} + A = -\frac{\kappa^2}{3} (\rho_0 + 3P_L^0), \quad (19)$$

where the dot stands for the derivative with respect to the proper time τ , $H_0 = \frac{\dot{a}_0}{a_0}$, $\rho_0 = \rho(t, y = 0)$, $P_L^0 = P_L(t, y = 0)$, and $P_T^0 = P_T(t, y = 0)$. For the sake of brevity, we have defined $A = K_{\tau\tau} K_{\tau\tau} - R_{tyty}|_{\Sigma_0}$, $B = K_{\chi\chi}^2 - R_{\chi y \chi y}|_{\Sigma_0}$, and without loss of generality $n(t, 0) = 1$. Solving the ty and yy components of the Einstein equations for $K_{\chi\chi}$ and $K_{\tau\tau}$, and performing the time integration we obtain

$$K_{\chi\chi}|_{\Sigma_0}^w = a_0 \sqrt{\dot{a}_0^2 + \frac{2\kappa^2}{3} \frac{\tilde{P}_T}{a_0^2} + k + \frac{E}{a_0^2}}, \quad (20)$$

$$K_{\tau\tau}|_{\Sigma_0}^w = \frac{\frac{2\kappa^2}{3} \frac{\tilde{P}_T}{a_0^4} - \frac{\kappa^2}{3} P_T^0 - \frac{\ddot{a}_0}{a_0} + \frac{E}{a_0^4}}{\sqrt{H_0^2 + \frac{2\kappa^2}{3} \frac{\tilde{P}_T}{a_0^4} + \frac{k}{a_0^2} + \frac{E}{a_0^4}}}, \quad (21)$$

where $E > 0$ is an integration constant, and

$$\tilde{P}_T \equiv \int_0^\tau P_T^0 a_0^4 H_0 d\tau. \quad (22)$$

From tt component of the Einstein equations one can quickly read

$$B = (K_{\chi\chi}^2 - R_{\chi y \chi y})|_{\Sigma_0}^w = H_0^2 a_0^2 + k - \frac{\kappa^2}{3} \rho a_0^2. \quad (23)$$

Furthermore, substituting the expressions (20) and (21) into $\chi\chi$ component of the Einstein equations yields the following expression for A

$$A = K_{\tau\tau}^2 - R_{tyty}|_{\Sigma_0}^w = -K_{\tau\tau}^2|_{\Sigma_0}^w - \frac{2\kappa^2}{3} \left(\rho_0 + \frac{3}{2} P_L^0 - P_T^0 + \frac{3}{a_0^4} \tilde{P}_T \right) - \frac{3E}{a_0^4}. \quad (24)$$

Returning now to the Sch-AdS bulk spacetime (7), we note that the corresponding four velocity u^μ and the normal vector n^μ being evaluated on the thick brane's core Σ_0 are, respectively

$$u^\mu|_{\Sigma_0}^\pm = (\dot{T}, \dot{a}, 0, 0, 0)|_{\Sigma_0}, \quad n^\mu|_{\Sigma_0}^\pm = \varepsilon_\pm (f^{-1} \dot{a}, f \dot{T}, 0, 0, 0)|_{\Sigma_0}, \quad (25)$$

where the sign function $\varepsilon = \pm 1$ takes care of the different patches of the Sch-AdS spacetime which might be glued to the brane, and $\dot{T}|_{\Sigma_0} = \frac{\sqrt{f_0 + \dot{a}_0^2}}{f_0}$. Subsequently, the relevant components of the extrinsic curvature tensor on Σ_0 computed from the Sch-AdS metric (7) are

$$K_{\chi\chi}|_{\Sigma_0}^{\pm} = \varepsilon_{\pm} a_0 \sqrt{f_0 + \dot{a}_0^2}, \quad (26)$$

$$K_{\tau\tau}|_{\Sigma_0}^{\pm} = -\frac{\varepsilon_{\pm}}{\sqrt{f_0 + \dot{a}_0^2}} \left(\ddot{a}_0 - \frac{\Lambda}{6} a_0 + \frac{C}{a_0^3} \right), \quad (27)$$

where $f_0 = f(r = a_0(\tau))$. The assumption of the Z_2 symmetry, with $y = 0$ as a fixed point, leads to $K_{ab}|_{\Sigma_0}^+ = -K_{ab}|_{\Sigma_0}^-$, implying that for matching of two interior patches of the Sch-AdS spacetime one has to choose $\varepsilon_+ = -1$ and $\varepsilon_- = +1$. This choice of the patches of the Sch-AdS metric is necessary to have the Z_2 symmetry which makes the problem simpler.

Moreover, from the metric (7), the nonzero components of the Riemannian curvature tensor on Σ_0 are calculated to be

$$R_{\chi r \chi r}|_{\Sigma_0}^w = \frac{1}{f_0} \left(\frac{\Lambda a_0^2}{6} - \frac{C}{a_0^2} \right), \quad R_{\chi T \chi T}|_{\Sigma_0}^w = f_0 \left(-\frac{\Lambda a_0^2}{6} + \frac{C}{a_0^2} \right), \quad R_{T r T r}|_{\Sigma_0}^w = \frac{\Lambda}{6} - \frac{3C}{a_0^4}. \quad (28)$$

We are now ready to write down explicitly the dynamical equations of the brane using the master equation (6).

4 Generalized Friedmann Equations

Substituting Eqs. (23), (25), (26), and (28) into the $\chi\chi$ component of the equation (6) we obtain

$$\sqrt{f_0 + \dot{a}_0^2} = \frac{w}{a_0} \left(-\frac{\Lambda}{3} a_0^2 + \frac{\kappa^2}{3} \rho_0 a_0^2 \right). \quad (29)$$

Taking the square of Eq. (29), substituting the expression (8) and rearranging, we arrive at the following equation

$$H_0^2 + \frac{k}{a_0^2} = \frac{2\kappa^2 w^2 (-\Lambda)}{9} \rho_0 + \frac{\kappa^4 w^2}{9} \rho_0^2 + \left(\frac{\Lambda}{6} + \frac{w^2 \Lambda^2}{9} \right) + \frac{C}{a_0^4}. \quad (30)$$

Assuming the brane energy density profile as a Taylor series around $y = 0$

$$\rho(t, y) = \rho_0 + y \frac{\partial \rho}{\partial y} \Big|_{y=0} + \frac{1}{2} y^2 \frac{\partial^2 \rho}{\partial y^2} \Big|_{y=0} + O(y^3), \quad (31)$$

we conveniently define an effective four-dimensional energy density ϱ associated to the five-dimensional energy density ρ as

$$\varrho = \int_{-w}^w \rho dy \simeq 2w \rho_0 + O(w^2). \quad (32)$$

We then identify

$$\frac{\Lambda_4}{3} = \frac{\Lambda}{6} + \frac{w^2 \Lambda^2}{9}, \quad (33)$$

$$8\pi G = \frac{\kappa^2 w(-\Lambda)}{3}. \quad (34)$$

Putting all this together, Eq. (30) turns into the following form

$$H_0^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3} \varrho + \frac{\kappa^4}{36} \varrho^2 + \frac{\Lambda_4}{3} + \frac{C}{a_0^4}. \quad (35)$$

This equation is our main result. As we see, there is a linear in addition to a quadratic term in the matter density, due to the non-vanishing of the thickness w , which is a novel effect. There is no need of introducing an ad hoc tension for the brane, and splitting it from the matter density on the brane. According to this equation the cosmological expansion undergoes a transition from a high energy regime $\kappa^2 \varrho \gg w(-\Lambda)$, where the dominated ϱ^2 term yields the unconventional cosmological expansion, into a low energy regime $\kappa^2 \varrho \ll w(-\Lambda)$ where the brane observers recover the standard cosmology described by the usual Friedmann equation. The compatibility with the Big Bang Nucleosynthesis (BBN) puts an essential constraint on the parameters of the model, so that to preserve the predictions of standard cosmology, the high energy regime, where the ϱ^2 term is significant, must occur before BBN era. From Eqs. (34) and (35) this implies that $\kappa^{-2} w |\Lambda| \geq (1 \text{ MeV})^4$, yielding the constraint $M \geq 10^4 \text{ GeV}$, for the fundamental mass scale defined by $\kappa^2 = M^{-3}$, the same result as obtained in the thin brane case.

Taking the thickness of the brane equal to the curvature size of the AdS defined as $\Lambda = \frac{-6}{l^2}$, we obtain from (33) exactly $\Lambda_4 = 0$. This means that the effective 4-dimensional cosmological constant induced on the brane vanishes up to the third order in the thickness. A residual term proportional to the third order may still remain. This may be a hint to the solution of the cosmological constant problem!

The effective brane cosmological constant has an interesting behavior too. According to (33), it is up-lifted relative to its value in the AdS bulk. This effect is similar to the KKLT up-lifting of the AdS minimum by inclusion of $\overline{D3}$ branes in the warped geometry put forward in [23], derived from purely geometrical considerations. If there is any deep connection to the KKLT uplifting effect remains to be seen.

Assuming again $2w = l$, from (34), the 4-dimensional gravitational constant then becomes $8\pi G = \frac{\kappa^2}{l}$. This is exactly the value derived by the dimensional compactification of the fifth dimension. Another consequence worth mentioning is the proportionality of the four-dimensional Newton's constant given by (34) to the brane thickness. Assuming a time dependent brane thickness, this induces a time evolution for the Newton's constant. On the cosmological scale this is experimentally constrained [24], imposing tight restrictions on the time dependence of the brane thickness.

The thin brane limit of our thick brane is also easily derived. In this limit the Eq. (35) reduces to the unconventional Friedmann equation of thin brane cosmology [17]

$$H_0^2 + \frac{k}{a_0^2} = \frac{\kappa^4}{36} \varrho^2 + \frac{\Lambda}{6} + \frac{C}{a_0^4}, \quad (36)$$

We may also look at the acceleration equation and its thin brane limit. Take the expressions (24), (25), (27), and (28) to write down the $\tau\tau$ component of Eq. (6) explicitly. We then end

up with the following equation:

$$\frac{\frac{\ddot{a}_0}{a_0} - \frac{\Lambda}{6} + \frac{C}{a_0^4}}{\sqrt{f_0 + \dot{a}_0^2}} = \frac{-w}{a_0} \left(\frac{2\kappa^2}{3} \left(\rho_0 + \frac{3}{2}P_L^0 - P_T^0 + \frac{3}{a_0^4}\tilde{P}_T \right) + \frac{\Lambda}{6} + \frac{3E}{a_0^4} + \frac{3C}{a_0^4} \right), \quad (37)$$

in deriving (37), we see from Eqs. (2) and (4) that $K_{\tau\tau}|_{\Sigma_0}^+ - K_{\tau\tau}|_{\Sigma_0}^w = O(w)$, and can therefore be neglected.

Now, defining the four-dimensional effective quantities associated to the five-dimensional longitudinal and transverse pressures P_L and P_T in the form

$$p_L = \int_{-w}^{+w} P_L dy \simeq 2wP_L^0 + O(w^2), \quad (38)$$

$$p_T = \int_{-w}^{+w} P_T dy \simeq 2wP_T^0 + O(w^2), \quad (39)$$

we realize that in the zero thickness limit ($w \rightarrow 0$), Eq. (37) reduces to the Raychaudhuri equation for a thin brane [25]:

$$\frac{\ddot{a}_0}{a_0} + \frac{\dot{a}_0^2}{a_0^2} + \frac{k}{a_0^2} = -\frac{\kappa^4}{36}\varrho(\varrho + 3p_L) + \frac{\Lambda}{3}, \quad (40)$$

where we have used the thin brane equation (36) and the fact that the profile for the transverse pressure P_T will not blow up in the thin brane limit. Therefore, we get the familiar thin brane limit from our thick brane model, as would be expected.

Note now that the time component of the covariant derivative of the brane energy-momentum tensor (10), using the metric (9), leads to the familiar energy conservation condition on the core of the thick brane:

$$\dot{\rho}_0 + 3H_0(\rho_0 + P_L^0) = 0. \quad (41)$$

Let us assume the arbitrary effective equations of state of the form

$$p_L = \omega_L \varrho, \quad p_T = \omega_T \varrho, \quad (42)$$

with constants ω_L and ω_T . The conservation equation (41) can then be integrated with the result as usual

$$\rho_0 = \rho_i a_0^{-3(1+\omega_L)}, \quad (43)$$

where ρ_i is a constant. Then \tilde{P}_T , according to the definition (22), can be computed as

$$\tilde{P}_T = \frac{\rho_i \omega_T}{1 - 3\omega_L} a_0^{1-3\omega_L}. \quad (44)$$

Of great cosmological interest is the possibility of a late time accelerated expansion on the brane. To investigate this possibility, we take a closer look at the generalized acceleration equation (37). Inserting Eqs. (42), (43), and (44), Eq. (37) can be recast as

$$\begin{aligned} \frac{\ddot{a}_0}{a_0} &= -\frac{\sqrt{f_0 + \dot{a}_0^2}}{a_0} \frac{\kappa^2 \varrho}{3} \left(1 + \frac{3\omega_L}{2} - \omega_T + \frac{3\omega_T}{1 - 3\omega_L} \right) + \frac{1}{6} \left(\Lambda_4 + \frac{\Lambda}{2} \right) \\ &+ \frac{\kappa^2 \varrho w}{6} \left(\frac{-\Lambda}{6} \right) - \frac{3w\sqrt{f_0 + \dot{a}_0^2}}{a_0} \left(\frac{E}{a_0^4} + \frac{C}{a_0^4} \right) - \frac{C}{a_0^4}, \end{aligned} \quad (45)$$

where we have used Eqs. (29) and (33). Note that at low energies, i.e. at late times, the two last terms on the right hand side of Eq. (45) redshift quickly and one can then neglect them. Hence, this equation tells us that for the acceleration term on the left hand side of Eq. (45) to be positive one or both of the following conditions must be satisfied

$$1 + \frac{3\omega_L}{2} - \omega_T + \frac{3\omega_T}{1 - 3\omega_L} < 0. \quad (46)$$

$$\Lambda_4 > \frac{-\Lambda}{2} \quad (47)$$

In particular, in the case of $\omega_L = 0$ for dust matter, the constraint (46) immediately reduces to $\omega_T < -\frac{1}{2}$. Consequently, we see that an accelerated cosmological expansion on the core of the thick brane is possible if one includes the matter having a negative pressure along the extra dimension in the brane energy-momentum tensor, or if the effective 4-dimensional cosmological constant is positive, according to the condition (47) (see also [18]).

5 Conclusion

We have obtained a general equation (6) to study the dynamics of codimension one brane of finite thickness immersed in an arbitrary bulk spacetime. This was obtained in a general setting by imposing the Darmois junction conditions on the brane boundaries with the two embedding spacetimes and then using an expansion scheme for the extrinsic curvature tensor at the brane boundaries in terms of the proper thickness of the brane. Our formalism is valid for any brane whose thickness is small compared to its curvature radius. Using this approach we gave the generalized Friedmann equations written for expansion of the extrinsic curvatures up to the first order of the brane proper thickness governing the cosmological evolution of the core of a thick brane embedded in a five-dimensional Schwarzschild Anti-de Sitter spacetime with a Z_2 symmetry. The derived equations for the thick brane have the well-known limit of the thin brane equations.

There are some novel effects for this finite thick brane cosmology. First, the generalized Friedmann equation shows a linear in addition to a quadratic term in the density. Therefore, the late time behavior is the same as the standard cosmology without introducing an ad hoc brane tension into the energy-momentum tensor of the brane. Second, the effective induced 4-dimensional cosmological constant of the brane is increased similar to the KKLT uplifting of the AdS minimum. It turns out that this 4-dimensional cosmological constant vanishes for a thickness equal to the AdS curvature size, up to the third power of the thickness. The 4-dimensional Newton's gravitational constant is then equal to the 5-dimensional one divided by the AdS length, which is similar to the result derived through the dimensional compactification. According to the equation (45), the universe filled with dust matter, will be accelerating at late times, if the pressure along the extra dimension in the brane energy-momentum tensor is negative, or if the effective cosmological constant satisfies the condition (47).

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7 Appendix

The non-trivial components of the five-dimensional Einstein tensor $G_{\mu\nu}$ for the metric (9) are computed as

$$G_{00} = 3\frac{\dot{a}^2}{a} - 3n^2\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right) + 3k\frac{n^2}{a^2}, \quad (48)$$

$$G_{ij} = a^2\gamma_{ij}\left(2\frac{a''}{a} + \frac{n''}{n} + \frac{a'^2}{a^2} + 2\frac{a'n'}{n}\right) + \frac{a^2}{n^2}\gamma_{ij}\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}\dot{n}}{an}\right) - k\gamma_{ij}, \quad (49)$$

$$G_{0y} = 3\left(\frac{n'\dot{a}}{n a} - \frac{\dot{a}'}{a}\right), \quad (50)$$

$$G_{yy} = 3\left(\frac{a'^2}{a^2} + \frac{a'n'}{an}\right) - \frac{3}{n^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{n}}{a n}\right) - 3\frac{k}{a^2}, \quad (51)$$

where a dot stands for a derivative with respect to t and a prime a derivative with respect to y .

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